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SURFACE TENSION DRIVEN INSTABILITY OF A LIQUID FILM FLOW DOWN A HEATED INCLINE

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Abstract—The onset of surface tension driven stationary circulation in a thin film flow down a heated incline is studied by use of linear theory. In the limiting case of a horizontal liquid film, the onset of circulation takes place at finite critical wave numbers and Marangoni numbers for a large range of relevant flow parameters if the stabilizing effect of gravity neglected by previous workers is taken into account. If the angle of inclination Φ of the incline is so small that $\sin\Phi \ll 1$ and if the Rayleigh number is of order $\sin^3\Phi$, then the flow structure corresponding to the neutrally stable state is that of stream-wise oriented vorticies. The critical Marangoni number of the film on a heated incline is smaller than that of the corresponding horizontal film by an amount $\sin^2\Phi F$ where F depends on relevant flow parameters.

NOMENCLATURE

C_p, heat capacity;
d, film thickness;
h, $(1-y^2)/2$;
k, thermal conductivity;
p, dimensionless pressure $= P/(\rho_m \kappa_m^2/d^2)$;
Pr, Prandtl number $= \nu_m/\kappa_m$;
q, heat-transfer coefficient;
S, surface tension;
t, time;
T, temperature;
U, velocity component in X direction;
U', V', W', perturbation velocity components in X, Y, Z directions;
u, v, w, dimensionless perturbation velocity components $= (U'V'W')/(\kappa_m/d)$;
$\bar{U}_*$, dimensionless steady state velocity, in x-direction;
V, velocity vector;
X, Y, Z, dimensional cartesian coordinates;
x, y, z, dimensionless cartesian coordinates $= X/d$, Y/d, Z/d.

Greek symbols

γ, isobaric thermal expansion coefficient $= -(1/\rho)(\partial\rho/\partial T)_p$;
ΔT, a temperature difference in the liquid $= T_w - T_m$;
η, dimensionless free surface deflexion $= B/d$;
Θ, dimensionless temperature perturbation $= T'/\Delta T$;
κ, thermometric conductivity;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density;
τ, dimensionless time $= t(d^2/\kappa_m)$;
Φ, angle of inclination of the film to the horizontal;
Ω, frequency of the disturbance.

Dimensionless groups

C, Crispation number $= Sd/\mu_m\kappa_m$;
Pe, Péclét number $= gd^3/\nu_m\kappa_m = WeC$;
M, Marangoni number $= -S_T\Delta T/\mu_m\kappa_m$;
Bi, Biot number $= qd/k_m$;
Ra, Rayleigh number $= gd^3\gamma\Delta T/\nu_m\kappa_m = Pe\gamma\Delta T$;
We, Weber number $= \rho_m g d^2/S$.

Superscripts

$\bar{}$, steady state quantities;
$'$, disturbance quantities.

Subscripts:

c, critical value;
i, imaginary part;
m, properties evaluated at the mid-depth of the liquid layer;
w, properties evaluated at wall.

1. INTRODUCTION

THE ONSET of the buoyancy-driven convection in a shallow pool of liquid steadily heated from below has been successfully predicted by use of linear theory [1]. The possible flows corresponding to the neutrally stable state in a liquid layer of unbounded horizontal extent form an infinitely degenerate set according to linear theory. The degeneracy can be eliminated by taking into account the nonlinear effects [2–8] or the anisotropy imposed by particular boundary geometries [5, 9–12] and the primary flows [13–18]. The theoretical predictions are generally in good agreement with experimental observations, and thus our understanding of the buoyancy-driven instability is relatively complete.

The study in the surface-tension-driven instability of a thin liquid layer is less extensive. Block [19], Pearson [20] and Sternling and Scriven [21] demonstrated that the surface tension gradient may

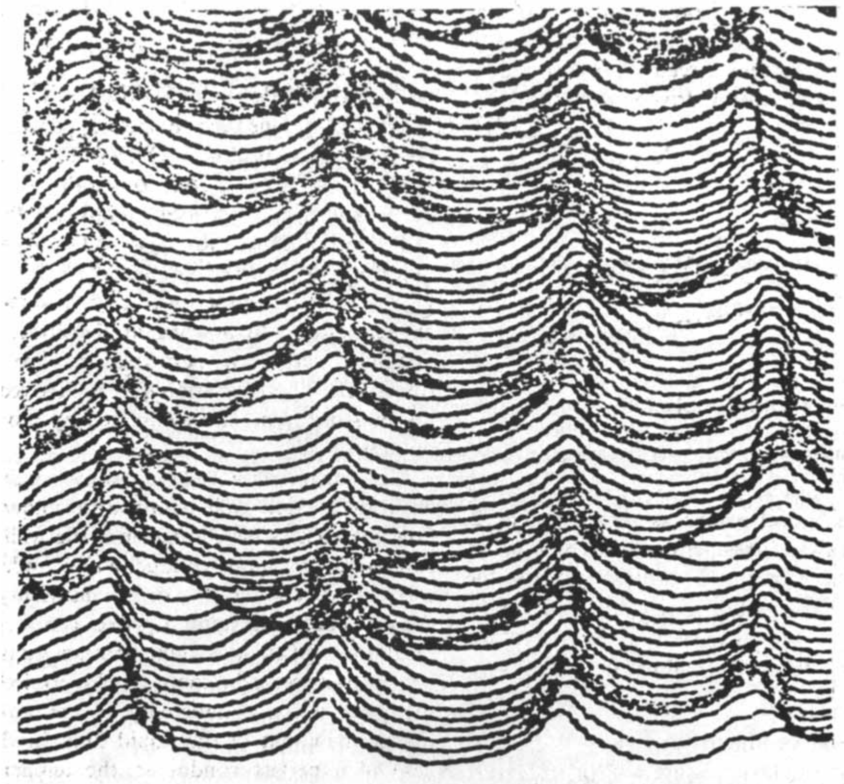


FIG. 1. Surface micrograph of a photographic film exhibiting surface tension driven instability. Each curve in the figure is obtained from tracing the elevation of a photographic film along a given horizontal line. The magnification factor for the elevation is 2000 and that for the horizontal distance is 50.

cause the instability. In Pearson's original analysis, the free surface was assumed to be extendable but inflexible. He invoked Newton's law of heat transfer at the interface and predicted the onset of stationary circulation in a horizontal liquid pool at finite critical flow parameters. Scriven and Sternling [22] extended Pearson's analysis to allow the deformation of a free surface. In ignoring gravity waves, they obtained results which suggested that the pool was always unstable to disturbances of very large wavelength. It was this peculiar prediction concerning long wavelength disturbances that motivated Smith's [23] analysis in which gravity waves were included. By use of Pearson's free surface boundary condition, Nield [24] demonstrated that the surface tension gradient and buoyancy can enhance each other in causing instability. However, all these analyses do not predict the onset of circulation in a film cooled from the bottom observed by Block.

Smith's numerical computations suggested that the cellular circulation may occur in a two-layered film cooled from the bottom. Smith's analysis was extended by Zeren and Reynolds [25] who included the effect of buoyancy. Their theoretical predictions do not compare well with experiments. The disagreement between theories and experiments is susually attributed to the presence of surfactants which are extremely difficult to remove. The stabilizing effect of surfactants has been shown by Berg and Acrivos [26] and Berg, Boudart and Acrivos [27]. Nield's prediction for the onset of circulation in a liquid pool heated from below enjoys exceptionally good agreement with the careful experiments of Palmer and Berg [28]. Despite its inability to predict instability in a film cooled from below, the Pearson–Nield model seems to be adequate for the case of a film heated from below. The linear theory again predicts an infinitely degenerate set of possible flows subsequent to the onset of surface-tension-driven instability. The discussions on the post instability flow structures for this case are relatively few in comparison with the case of buoyancy-driven instability. Scanlon and Segel [29] have shown that nonlinears effects in surface tension induced instability promotes the formation of hexagonal cells. Gumerman and Homsy [30] have indicated that the instability in concurrent two phase flow can take three possible forms: streamwise oriented roll vortices, long interfacial waves, and short Tollmein–Schlichting waves.

In this work, the onset of the surface tension gradient driven stationary circulation in a thin film flow down a heated inclined plane is studied by use of linear theory. The flow structure corresponding to the marginally stable state is examined. The condition under which the principle of exchange of stability is valid is investigated. The main results obtained in the subsequent analysis are summarized in the Abstract.

The problem considered in this study is of great importance in modern precision coating technology. Figure 1 shows a surface micrograph of a photographic film which exhibits a regular surface pattern created by the surface tension driven instability.

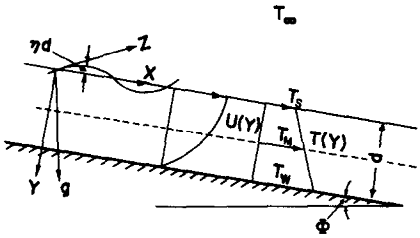


FIG. 2. Definition sketch.

2. STABILITY ANALYSIS AND RESULTS

Consider the stability of a liquid film flowing steadily down a heated incline as shown in Fig. 2. The steady fields of temperature T, density ρ, pressure P, and velocity $\mathbf{V}$ which satisfies the equations of Navier–Stokes, energy, continuity, linear state relation between density and temperature, and the constant wall temperature boundary conditions are

$$\bar{T}(y) = T_m - \Delta T(1-2y), \quad y = Y/d,$$

$$\bar{\rho}(y) = \rho_m[1+\gamma\Delta T(1-2y)], \quad \Delta T = T_w - T_m,$$

$$\bar{P}(y) = P_g + \rho_m g d \cos\Phi \cdot y[1+\gamma\Delta T(1-y)],$$

$$\bar{U}(y) = (gd^2/\nu_m)\sin\Phi\cdot[(1-y^2)/2 + \gamma\Delta T(1-3y^2+2y^3)/6], \quad (1)$$

where upper bars refer to the basic fields the stability of which is being considered, the subscript m denotes that the subscripted variables are to be evaluated at the mid-depth of the film, Φ is the angle of inclination of the bottom wall, $\mathbf{V}$ is the fluid velocity, Y is the normal distance measured from the unperturbed free surface into the film, d is the film thickness, T_w is the wall temperature, g is the gravitational acceleration, P_g is the ambient pressure, and γ is the thermal expansion coefficient. Note that the heat flux across each unit length of the film described by (1) is constant. To study the stability of the basic flow given in (1), we introduce the following perturbation quantities

$$\mathrm{T} = \bar{T}+T', \quad \rho = \bar{\rho}+\rho', \quad P = \bar{P}+P',$$

$$\mathbf{V} = (\bar{U}+U', V', W')$$

into the equations of mass, energy, state and momentum. Then using d, d^2/κ_m, κ_m/d, $\rho_m\kappa_m^2/d^2$ and ΔT as scaling quantities for length, time, velocity, pressure and temperature respectively, we have, after some manipulation, the following dimensionless governing equations of linear stability

$$L_1(\nabla^2 v) + v_x D^2\bar{U}_* = PrRa(\cos\Phi\cdot\nabla_1^2\Theta - \sin\Phi\cdot\Theta_{xy}) \quad (2)$$

$$L_2\Theta = 2v \quad (3)$$

$$\nabla_1^2 p = -L_1 v_y - v_x D\bar{U}_* - PrRa\sin\Phi\Theta_x \quad (4)$$

$$L_1(u_z - w_x) - v_z D\bar{U}_* = PrRa\sin\Phi\Theta_z \quad (5)$$

$$u_x + v_y + w_z = 0 \quad (6)$$

$\bar{U}_* = \sin\Phi[Pe(1-y^2)/2 + Ra(1-3y^2+2y^3)/6]$, where κ_m is the thermometric conductivity, ∇^2 the Laplacian, $(u,v,w) = (U',V',W')/(\kappa_m/d)$, $(x,y,z) = (X,Y,X)/d$, $\Theta = T'/\Delta T$, $D = d/dy$, $\nabla_1^2 = \nabla^2 - D^2$, $Pe = gd^3/\nu_m\kappa_m$ = Péclét number, $Pr = \nu_m/\kappa_m$ = Prandtl number, $Ra = gd^3\gamma\Delta T/\nu_m\kappa_m \equiv$ Rayleigh number, $L_1 = Pr\nabla^2 - \partial_\tau - \bar{U}_*\partial_x$, $L_2 = \nabla^2 - \partial_\tau - \bar{U}_*\partial_x$, and the subscripts, x, y, z denote partial differentiations. In arriving at the above equations nonlinear terms in perturbations are neglected, Boussinesq approximation [1] is invoked, some of the pressure terms are eliminated by cross differentiations [1, 17].

It is observed that the stability conditions vary even for a given wall temperature under the same flow parameters depending on if the wall is a good conductor or a poor conductor. To understand the effect of wall conductivity on the instability, we study the two limiting cases of a perfect conductor and an insulator. When the film becomes unstable, the temperature perturbation is quickly ironed out along the plate if the conductivity of the wall is much higher than that of the liquid film. In the limiting case of a perfect conductor, the temperature perturbation along the wall vanishes, i.e. at $y = 1$. $\Theta = 0$ for a perfect conductor (isothermal case). In the other case of an extremely poor conductor, the temperature perturbation in the film hardly has enough time to alter the base flow heat flux at the wall at the onset of instability because of its extremely poor conductivity. Thus, we have at $y = 1$

$\Theta_y = 0$ for an extremely poor conductor (constant heat flux case).†

The kinematic boundary condition at $y = 1$ is

$$v = v_y = 0.$$

The boundary conditions at the free surface $y = \eta$ can be expanded about $y = 0$ by use of the Taylor series expansions. Retaining only the linear terms in perturbations, we have:

kinematic condition,

$$v = \eta_\tau + U_*\eta_x,$$

tangential force balance

$$(\nabla_1^2 - D^2)v + \eta_x D^2\bar{U}_* - M\nabla_1^2(\Theta+2\eta) = 0,$$

normal forces balance

$$p = PrC\nabla_1^2\eta + 2Prv_y - PrPe(1+\gamma\Delta T)\eta\cos\Phi, \quad (7)$$

heat flux balance

$$D\Theta = Bi(\Theta+2\eta),$$

†Professor W. J. Minkowycz has pointed out that while this condition may be adequate for linear stability analysis of an extreme limiting case, it is hard to realize in experiments.

where M, C, and Bi are respectively the Marangoni, Crispation, and Biot numbers defined as:

$$M = -(\partial_T S)(\Delta T/\mu_m \kappa_m),\ C = Sd/\mu_m \kappa_m,\ Bi = qd/k_m,$$

S and q being respectively the surface tension and the heat-transfer coefficient. The pressure in the above equation is given by (4) in terms of v and Θ. The derivation of the free surface boundary conditions are given in the Appendix.

Note that v, θ and η are decoupled in the above differential system. Therefore only (2) and (3) need to be solved with the above boundary conditions in the stability analysis. If more detail of the perturbed flow field is desired, p, u and w can be obtained from (4)–(6) after v and Θ are determined.

The normal mode solution to the above formulated eigenvalue problem is sought in the form

$$[v, \Theta, p, \eta] = [\psi(y), \theta(y), \pi(y), \xi] \exp[i(\alpha x + \beta z + \Omega\tau)],$$

where α and β are the numbers of waves over a distance $2\pi d$ in the x and y directions respectively, and Ω is the wave frequency. Substituting this form into (2), (3) and their boundary conditions, we have the governing differential equations

$$\begin{gathered}[Pr(D^2-a^2) - i(\Omega + \alpha\bar{U}_*)](D^2-a^2)\psi + i\alpha D^2\bar{U}_* \\ = PrRa \\ (a^2\theta\cos\Phi + i\alpha\sin\Phi\, D\theta) \\ (D^2-a^2)\theta - i(\Omega+\alpha\bar{U}_*)\theta = 2\psi \qquad (8)\end{gathered}$$

and the boundary conditions

$$\psi = D\psi = 0,$$
$$\theta = 0 \text{ or } D\theta = O \text{ at } y = 1,$$

and

$$\begin{gathered}\psi = i(\Omega + a\bar{U}_*)\xi \\ (D^2+a^2)\psi - a^2M(\theta+2\xi) - i\alpha D^2\bar{U}_*\xi = 0, \\ Pr(D^2-3a^2)D\psi + a^2\,Pr[a^2C + \cos\Phi(Pe+Ra)]\xi \\ + i[\alpha D^2\bar{U}_*\psi - (\Omega+\alpha\bar{U}_*)D\psi + \alpha Pr\,Ra\,\mathrm{Sin}\Phi\theta] = 0, \\ D\Theta = Bi(\theta+2\xi) \quad \text{at} \quad y = 0,\end{gathered}$$

where $a = (\alpha^2+\beta^2)^{1/2}$.

Consider the case of small Φ and small d such that $\sin\Phi = \varepsilon \ll 1$, $Ra = O(\varepsilon^3)$. Expand the eignefunctions, the eigenvalue M and Ω in powers of ε

$$(\psi, \Theta, \xi, \Omega, M) = \sum_{k=0}^{n} \varepsilon^k[\psi_k(y), \theta_k(y), \xi_k, \Omega_k, M_k].$$

The remaining flow parameters α, β, Pr, Bi, C, Ra and Pe are considered independent parameters.

Substituting the series solution into (8) and its boundary conditions and collecting terms of $O(1)$, we have

$$(D^2-a^2)^2\psi_0 - (i\Omega_0/Pr)(D^2-a^2)\psi_0 = 0, \qquad (9)$$

$$(D^2-a^2)\theta_0 - i\Omega_0\theta_0 = 2\psi_0, \qquad (10)$$

and the boundary conditions $\psi_0 = D\psi_0 = 0$, $\theta_0 = 0$ or $D\theta_0 = 0$ at $y = 1$; and at $y = 0$

$$\begin{gathered}\psi_0 = i\Omega_0\xi_0, \\ (D^2+a^2)\psi_0 - a^2M_0(\theta_0+2\xi_0) = 0, \\ (D^2-3a^2)D\psi_0 + a^2(a^2C+Pr)\xi_0 + i(\Omega_0/Pr)D\psi_0 = 0, \\ D\theta_0 = Bi(\theta_0+2\xi_0). \qquad (11)\end{gathered}$$

The above differential system can be readily reduced to a homogeneous system in only one dependent variable θ_0 by use of (10) and the last equation of (11). If the instability sets in as stationary cellular convection, the principle of exchange of stability holds [31] and $\Omega = 0$ on the neutral curve. The corresponding eigenfunction for both cases of constant wall temperature and constant heat flux can be written as

$$\theta_0 = (A_0 + A_1y + A_2y^2)\cosh(ay) + (B_0 + B_1y + B_2y^2)\sinh(ay),$$

where

$$\begin{gathered}A_0 = -[8a^3(a^2C+Pe)/(sc-a)]A_2, \\ A_1 = [(s^2+2a^2)/a(sc-a)]A_2, \\ B_0 = [4Bi/aM_0 + (s^2-2a^2)/a^2(sc-a)]A_2, \\ B_1 = -A_2/a, \quad B_2 = -[s^2/(sc-a)]A_2, \\ c = \cosh(ay), \quad s = \sinh(ay).\end{gathered}$$

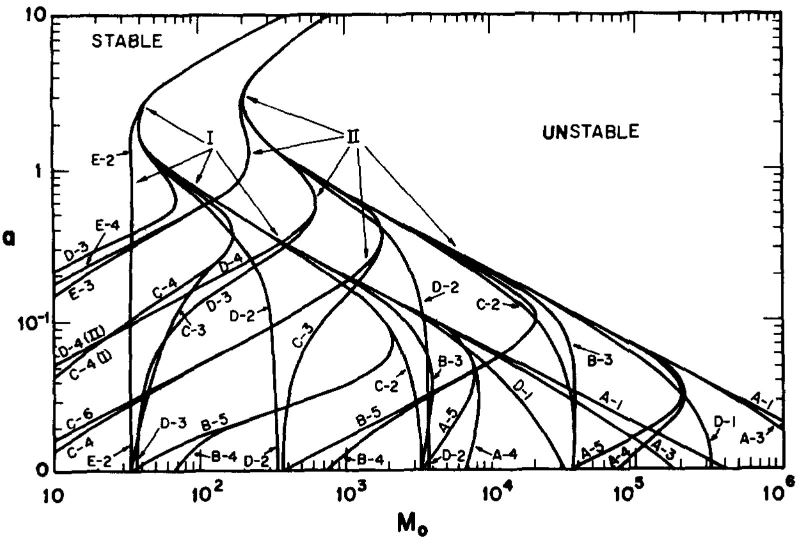
Note that the eigenfunction is here determined only up to an arbitrary multiplication factor as is expected for a linear homogeneous system. We now normalize the eigenfunction with the coefficient of $y^2\cosh(ay)$. Thus we put $A_2 = 1$ and set the coefficients of $y^2\cosh(ay)$ in the higher order solutions to zero. Note also that the above coefficients are determined without invoking the thermal boundary conditions at the rigid wall. Invoking this condition we have the eigenvalue on the neutral curve

$$M_0 = \frac{4a(\sinh a\cosh a - a)(Bi\sinh a + a\cosh a)}{(\sinh^3 a - a^3\cosh a) + 8a^5\cosh a/(a^2C+Pe)}$$

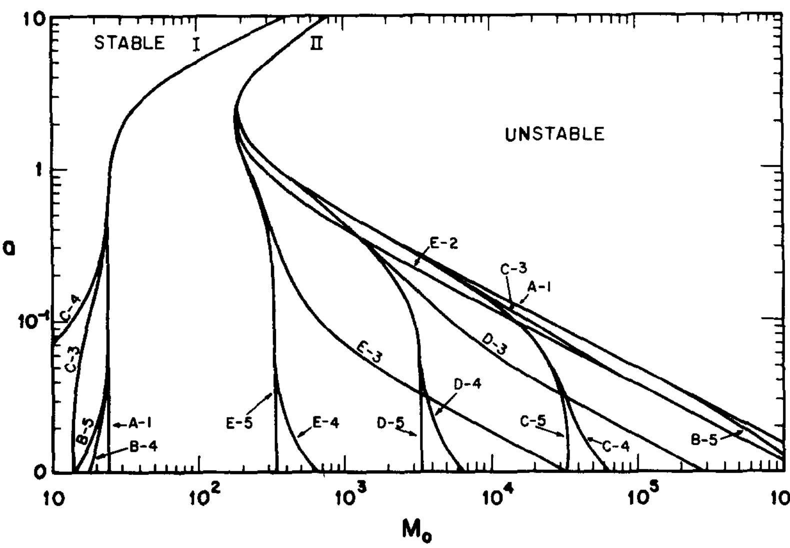
for the case of constant wall temperature, and

$$M_0 = \frac{4a(\sinh a\cosh a - a)(Bi\cosh a + a\sinh a)}{\cosh a(a^2+\sinh^2 a) - a\sinh a(a^2+2) + 8a^5\sinh a/(a^2C+Pe)}$$

for the case of constant wall heat flux.



3(a). Constant wall temperature case.



3(b). Constant heat flux case.

FIG. 3. Neutral stability curves. The regions on the left and the right of each curve are respectively the region of stability and instability. The set of curves for which $Bi = 0$ is denoted by I. The set of curves for which $Bi = 10$ is denoted by II. The letters A, B, C, D and E stand respectively for the values of the Crispation numbers 10^8, 10^6, 10^4, 10^3 and 10^2. The numbers 1, 2, 3, 4 and 5 on the curves designate respectively the values of the Weber numbers 10^2, 1, 10^{-2}, 10^{-4} and 10^{-6}.

Table 1. Asymptotic behavior of M_0 for zero wave number

	Constant temperature case	Constant heat flux case $Bi = 0$	Constant heat flux case $Bi \neq 0$
Finite C:	$C \cdot We(1+Bi)/3$	$24/[1+72/C \cdot We]$	∞
Finite C, $We = 0$:	0	0	$Bi/3$
$C \to \infty$:	∞	24	∞

The above results differ from those of Scriven and Sternling [22], if their surface viscosity is neglected, only by the term Pe which arises from the stabilizing effect of gravity at the free surface neglected by them. The inclusion of this term has a profound effect of rendering non trivial the critical Marangoni number and the critical wave number for a large range of other relevant parameters as is illustrated in Fig. 3 and Table 1. Note also that Pearson's results for the case of an inflexible free surface are recovered in the limit of $C \to \infty$ when the surface tension is infinite.

Similarly for the $O(\varepsilon)$ we have the governing differential equation at the marginal state

$$(D^2-a^2)^3\theta_1 = i\Omega_{1r}(1+1/Pr)(D^2-a^2)^2\theta_0+i\alpha Pe.$$
$$[(D^2-a^2)^2(h\theta_0)-D^2h(D^2-a^2)\theta_0 + (h/Pr)(D^2-a^2)^2\theta_0].$$

The boundary conditions at $y = 1$ are

$$\theta_1 = 0,$$
$$(D^2-a^2)\theta_1 = 0,$$
$$D(D^2-a^2)\theta_1 = i\Omega_{1r}D\theta_0$$

for the case of constant wall temperature, and

$$D\theta_1 = 0,$$
$$(D^2-a^2)\theta_1 = i\Omega_{1r}\theta_0,$$
$$D(D^2-a^2)\theta_1 = i\alpha PeDh\theta_0, \quad h = (1-y^2)/2$$

for the case of constant wall heat flux. The boundary conditions at the free surface are

$$(D^2-a^2)\theta_1 = i(\Omega_{1r}+\alpha Peh)D\theta_0/Bi$$
$$(D^4-a^4)\theta_1-2a^2(M_0/Bi)D\theta_1 = 2a^2(M_1/Bi)D\theta_0 + 2i\Omega_{1r}a^2+i\alpha Pe[2a^2h\theta_0+D^2hD\theta_0/Bi],$$
$$(D^2-3a^2)(D^2-a^2)D\theta_1 + (a^2/Bi)(a^2C+Pe)(D\theta_1-Bi\theta_1) = i(\Omega_{1r}+\alpha Peh)[(D^2-3a^2)D\theta_0 + (D^2-a^2)D\theta_0/Pr] + 3i\alpha PeD^2hD\theta_0.$$

The eigenfunction θ_1 can be written as

$$\theta_1(y) = i\Omega_{1r}\theta_{12}(y)+i\alpha Pe\theta_{13}(y)$$
$$\theta_{1j}(y) = \sum_{n=0}^{5} y^n[C_{nj}\cosh(ay) + S_{nj}\sinh(ay)], \quad j = 2, 3$$

where

$$C_{32} = (1+1/Pr)B_2/6a, \quad S_{32} = (1+1/Pr)A_2/6a,$$
$$C_{42} = C_{52} = S_{42} = S_{52} = 0,$$
$$C_{33} = (3+1/Pr)A_1/24a^2-B_0/12a + [a^2(1+1/Pr)-(3+4/Pr)]B_2/12a^3,$$
$$S_{33} = (1+1/Pr)(2a^2-9)A_2/24a^3-A_0/12a,$$
$$C_{43} = (9+4/Pr)A_2/48a^2, \quad S_{43} = (3+2/Pr)B_2/24a^2-A_1/16a\ldots$$
$$C_{53} = -(6+1/Pr)B_2/120a, \quad S_{53} = -(6+1/Pr)A_2/120a.$$

The remaining coefficients (C_{0j}, C_{1j}, C_{2j}) and (S_{0j}, S_{1j}, S_{2j}) belong to the complementary solution. $C_{2j} = 0$ according to our eigenfunction normalization described earlier. The rest of the integration constants are determined from the inhomogeneous boundary conditions excluding that of the free surface tangential force balance. The explicit expressions of the integration constants are lengthy but available [32], and therefore need not be given here. Substituting the eigenfunction thus obtained into the tangential force boundary condition, we obtain respectively from its real and imaginary parts

$$M_1 = 0 \quad \text{and} \quad \Omega_{1r} = \alpha Pe(Q_1/Q_2) \qquad (12)$$

where

$$Q_1 = 4A_2/M_0-a^2A_0+4a^3S_{13} + 24(C_{43}+aS_{33}) - a^2(M_0/Bi)(C_{13}+aS_{03}),$$
$$Q_2 = 2a^2A_0+a^2(M_0/Bi)(C_{12}+aS_{02}) - 4a^3S_{12}-24aS_{32}.$$

In general, $\Omega_{1r} \neq 0$ and thus the principle of exchange of stability does not hold for nonvanishing Φ even when it is valid for $\Phi = 0$. Moreover, the correction to M due to angle of inclination Φ must be sought in higher order solution, since $M_1 = 0$.

The governing equation for the $O(\varepsilon^2)$ problem is

$$\begin{aligned}(D^2-a^2)^3\theta_2 = {} & i\Omega_{2r}(1+1/Pr)(D^2-a^2)\theta_0 \\ & +(\Omega_{1r}^2/Pr)[(D^2-a^2)\theta_0 \\ & -(1+Pr)(D^2-a^2)^2\theta_{12}] \\ & +(\alpha\Omega_{1r}Pe/Pr)[2h(D^2-a^2)\theta_0 \\ & +2DhD\theta_0+D^2h(D^2-a^2)\theta_{12} \\ & -h(D^2-a^2)^2\theta_{12}-Pr(D^2-a^2)^2(h\theta_{12}) \\ & -(1+Pr)(D^2-a^2)^2\theta_{13}] \\ & +(\alpha^2Pe^2/Pr)[h^2(D^2-a^2)\theta_0 \\ & +2hDhD\theta_0-Pr(D^2-a^2)^2(h\theta_{13}) \\ & -h(D^2-a^2)^2\theta_{13} \\ & +D^2h(D^2-a^2)\theta_{13}].\end{aligned}$$

The boundary conditions at $y = 1$ are

$$\theta_2 = 0, \quad (D^2-a^2)\theta_2 = 0$$
$$D(D^2-a^2)\theta_2 = i\Omega_{2r}D\theta_0-\Omega_{1r}^2D\theta_{12}-\alpha\Omega_{1r}PeD\theta_{13}$$

for the case of constant wall temperature, and $D\theta_2 = 0$,

$$(D^2-a^2)\theta_{12} = i\Omega_{2r}\theta_0-\Omega_{1r}^2\theta_{12}-\alpha\Omega_{1r}Pe\theta_{13},$$
$$D(D^2-a^2)\theta_2 = -\alpha\Omega_{1r}Pe\theta_{12}Dh-\alpha^2Pe^2\theta_{13}Dh,$$

for the case of constant wall heat flux. The boundary conditions at $y = 0$ are

$$(D^2-a^2)\theta_2 = i\Omega_{2r}D\theta_0/Bi-\Omega_{1r}^2D\theta_{12}/Bi -\alpha\Omega_{1r}Pe(D\theta_{13}+hD\theta_{12})/Bi-\alpha^2Pe^2hD\theta_{13}/Bi,$$

$$\begin{aligned}(D^4-a^4)\theta_2-2a^2(M_0/Bi)D\theta_2 = {} & 2i\Omega_{2r}a^2\theta_0+2a^2M_2D\theta_0/Bi \\ & -\Omega_{1r}^2(D^2+a^2)\theta_{12}-\alpha\Omega_{1r}Pe[(D^2+a^2)\theta_{13} \\ & +h(D^2+a^2)\theta_{12}+D^2hD\theta_{12}/Bi] \\ & -\alpha^2Pe^2[h(D^2+a^2)\theta_{13}+D^2hD\theta_{13}/Bi],\end{aligned}$$

$$\begin{aligned}&(D^2-3a^2)(D^2-a^2)D\theta_2+(a^2/Bi)(a^2C+Pe)(D\theta_2 - Bi\theta_2) \\ &= i\Omega_{2r}[(D^2-3a^2)D\theta_0+(D^2-a^2)D\theta_0/Pr] \\ &+\Omega_{1r}^2[D\theta_0/Pr-(D^2-3a^2)D\theta_{12}-(D^2 - a^2)D\theta_{12}/Pr] \\ &+\alpha\Omega_{1r}Pe\{2hD\theta_0/Pr-[(D^2-3a^2)D\theta_{13}+h(D^2 - 3a^2)D\theta_{12}] \\ &-[(D^2-a^2)D\theta_{13}+h(D^2-a^2)D\theta_{12}]/Pr - 3D^2hD\theta_{12}\} \\ &+\alpha^2Pe^2\{h^2D\theta_0/Pr-h[(D^2-3a^2)D\theta_{13}+(D^2 - a^2)D\theta_{13}/Pr - 3D^2hD\theta_{13}]\}+a^2Pe(D\theta_0-Bi\theta_0)/Bi.\end{aligned}$$

The $O(\varepsilon^2)$ eigenfunction can be written in the form

$$\theta_2(y) = i\Omega_{2r}\theta_{21} + \Omega_{1r}^2\theta_{22} + \alpha\Omega_{1r}Pe\theta_{23} + \alpha^2 Pe^2\theta_{24} + \theta_{25}$$

where

$$\theta_{2k} = \sum_{r=o}^{8} y^r[f_{rk}\cosh(ay) + g_{rk}\sinh(ay)].$$

The determination of the eigenfunction and the eigenvalue follows exactly the same procedure described in the $O(\varepsilon)$ problem. f_{rk} and g_{rk} are lengthy expressions in terms of the flow parameters. They are available [32] and will not be given here. Substitution of the eigenfunction into the tangential force boundary condition gives

$$\Omega_{2r}[(D^4 - a^4)\theta_{21} - 2a^2\theta_0 - 2a^2(M_0/Bi)D\theta_{21}]_{y=0} = 0 \quad (13)$$

$$M_2 = \alpha^2 Pe^2 F_1(a, M_0, Bi, C, Pr, We) - F_2(a, M_0, Bi, C, Pr, We), \quad (14)$$

where $We \equiv Pe/C \equiv$ Weber number, and

$$F_1 = H_1(Q_1/Q_2)^2 + H_2(Q_1/Q_2) + H_3,$$

$$H_1 = (M_0/4)[(S_{12} + aC_{02})/a + 2ag_{12} + 12(f_{42} + ag_{32})/a^2] - (M_0^2/4Bi)(f_{12} + ag_{02})$$

$$H_2 = (M_0/4)[(S_{13} + aC_{03})/a + (S_{12} + aC_{02})/2a - C_{12} + aS_{02})/2a^2 Bi + 2ag_{13} + 12(f_{43} + ag_{33})/a^2 - (M_0^2/4Bi)(f_{13} + ag_{03})$$

$$H_3 = (M_0/4)[(S_{13} + aC_{03})/2a - (C_{13} + aS_{03})/2a^2 Bi + 2ag_{14} + 12(f_{44} + ag_{34})/a^2] - (M_0^2/Bi)(f_{14} + ag_{04})$$

$$F_2 = [Wea^3/CM_0^2(We + a^2)^2 \cdot (\sinh a \cosh a - a)] \times [1 + Bi(\tanh a)/a].$$

Thus, to $O(\varepsilon^2)$ we have

$$M = M_0 + \varepsilon^2[\alpha^2 Pe^2 F_1 - F_2].$$

It is seen from the expression of F_2 that F_2 is positive definite. Because of the complicated expression of F_1, we are not able to prove that F_1 is also positive definite. However, extensive numerical evaluation of F_1 shows that F_1 is positive around the critical Marangoni number M_{0c} and the critical wave number a_c if both of them are nontrivial. Typical numerical results are given in Tables 2, 3 and 4. It follows from the above expression of M that for given Bi, C, We, Pr, M is the minimum if $\alpha = 0$ for each value of a for which $F_1 > 0$. Thus onset of surface tension gradient driven instability sets in a film on a heated inclined plane as streamwise rolls over a wide range of flow parameters. These rolls are non-oscillatory to $O(\varepsilon^2)$ accuracy, since $\Omega_{1r} = 0$ at $\alpha = 0$ according to (12) and $\Omega_{2r} = 0$ according to (13) because the quantity in brackets in (13) is not identically zero. Note that the rolls are more easily observable for the case of constant heat flux at high Biot numbers (cf. Table 1). Note also that the critical Marangoni number of an inclined film is smaller than that of the corresponding horizontal film only by a small amount of $O(\varepsilon^2 F_2)$, since F_2 is positive definite.

The numerical results given in this section agree with those obtained from the orthogonality condition of the present differential system with its adjoint system [32].

Table 2. Isothermal rigid surface
$Bi = 10$, $C = 10^6$, $We = 10^{-2}$, $Pr = 10$

a	M_0	F_1	F_2
0.01	3.6723(4)*	−4.674(20)	1.803(4)
0.10	2.7526(4)	−1.737(10)	2.586(3)
0.50	1.8523(3)	4.506(6)	7.071(−2)
1.00	5.4074(2)	1.146(6)	4.090(−4)
2.00	2.3166(2)	3.028(5)	3.939(−6)
2.60	2.0742(2)	1.828(5)	8.040(−7)
2.70	2.0678(2)	1.697(5)	6.422(−7)
2.80	2.0683(2)	1.577(5)	5.174(−7)
4.00	2.4397(2)	6.946(4)	5.731(−8)
6.00	3.8600(2)	1.595(4)	2.288(−9)
8.00	5.7613(2)	3.373(3)	8.298(−11)
10.00	8.0000(2)	8.440(2)	2.638(−12)

Table 3. Constant heat flux rigid surface
$Bi = 10$, $C = 10^6$, $We = 10^{-2}$, $Pr = 10$

a	M_0	F_1	F_2
0.01	2.3831(6)	−8.2811(22)	8.351(3)
0.10	2.4034(4)	−3.238(9)	2.152(1)
0.50	1.0805(3)	5.428(7)	5.565(−3)
1.00	3.6352(3)	1.316(7)	1.127(−4)
2.00	1.9893(2)	2.643(6)	2.733(−6)
2.40	1.9167(2)	1.562(6)	1.041(−7)
2.50	1.9166(2)	1.374(6)	8.375(−7)
2.60	1.9223(2)	1.210(6)	6.785(−7)
4.00	2.4101(2)	2.151(5)	5.588(−8)
6.00	3.8577(2)	2.252(4)	2.285(−9)
8.00	5.7612(2)	3.587(3)	8.298(−11)
10.00	8.0000(2)	8.499(2)	2.638(−12)

Table 4. Isothermal case: results by orthogonality method
$Bi = 10^{-2}$, $C = 10^4$, $We = 10^{-6}$, $Pr = 10^{-3}$

a	M_0	F_1	F_2
0.1	−2.6165(3)	6.15161(0)	2.75057(−3)
0.5	−3.2893(2)	−1.11021(−1)	8.48167(−4)
1.0	1.2834(1)	3.39731(−3)	2.48237(−5)
2.0	3.1991(1)	7.50893(−3)	9.99198(−7)
2.6	3.8212(1)	7.96900(−3)	3.79224(−7)
2.8	4.1007(1)	8.11228(−3)	2.97045(−7)
4.0	6.6421(1)	8.16044(−3)	1.09115(−7)
6.0	1.4177(2)	4.50281(−3)	4.46844(−8)
8.0	2.5318(2)	1.45119(−3)	2.50060(−8)
10.0	3.9720(2)	4.33279(−4)	1.60001(−8)

3. DISCUSSION

It should be pointed out that F_1 is not positive, according to our numerical computation, when the

Prandtl number is extremely small or when the critical wave number is zero. For these cases the marginal state may not be the stationary convection rolls. The oscillatory mode may dominate the instability in a film of small Prandtl number [19, 21], and the neglected nonlinear effect may be dominatingly important when the wave number approaches zero.

The analysis given in Section 2 was carried out for the case of $\sin\Phi \ll 1$ and $Ra = O(\sin^3\Phi)$ but all other flow parameters need not be small. For this case, the forced convection acts as a higher order source term, because of the small velocity associated with the small Φ. However, it can be seen from the expression of $\bar{U}_*$ that the convective term may still remain small even if Φ is not small if $Pe = \varepsilon \ll 1$ and $Ra = O(Pe^3)$. Since $Pe = gd^3/\nu_m\kappa_m$, the latter case corresponds to the creeping flow associated with large viscosity and small film thickness. The stability analysis for this case can be carried out in exactly the same manner as that given in the previous section except the small parameter ε of expansion is now Pe and Φ must be treated as a free parameter. Such analysis is not yet available. It would be interesting to see whether the marginal state of the surface tension driven instability of a thin film on a heated incline is streamwise oriented or transverse rolls when the angle of inclination is near vertical. It is known that for the case of buoyancy-driven instability the neutral state is transverse rolls with their axis normal to the mean flow if Φ is near vertical [16, 17, 34, 35].

It should be pointed out that the present study is relevant only to a very thin film flow in which neither the gravity-capillary ripples nor the buoyancy-driven convection is the dominating mode of instability. The effects of surface tension variation on the ripple formation have been studied by Bankoff [36], Marschall and Lee [37] and Lin [38].

Quantitative experimental data for the present problem are not available to the authors, and comparisons between the theory and experiments cannot be made presently.

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APPENDIX

In deriving the conditions to be applied at the unknown perturbed free surface $Y = B(X, Z, t)$, we neglect the shear and inertial forces in the ambient gas and also the nonlinear quantities in perturbations.

Since the time rate of displacement in the Y-direction is by definition V', we have the kinematic condition

$$V' = DB/Dt. \qquad (A1)$$

In order for each element of the free surface to have finite accelerations in the X and Z directions, we must have

$$\mu_m(V'_X + \bar{U}_Y + U'_Y) + S_X = 0 \text{ and } \mu_m(V'_Z + W'_Y) + S_Z = 0. \qquad (A2)$$

Similarly, the force balance in the Y-direction requires

$$-(\bar{P} + P' - P_g) + 2\mu_m V'_Y + S(B_{XX} + B_{ZZ}) = 0. \qquad (A3)$$

By use of the Newton's heat-transfer law, the energy balance at the free surface can be written as

$$k_m(\bar{T}_Y + T'_Y) = q(\bar{T} + T' - T_g). \qquad (A4)$$

Expanding all variables depending on Y around $Y = 0$ by use of the Taylor series expansion, and retaining only linear terms in perturbations, and using the primary flow relations, the above boundary conditions at $Y = B$ can be reduced to the conditions at $Y = 0$:

$$V' = B_t + \bar{U}B_X,$$

$$\mu_m(V'_{XX} + V'_{ZZ} - V'_{YY} + B_X U_{YY}) + [T'_{XX} + T'_{ZZ} + T_Y(B_{XX} + B_{ZZ})] = 0,$$

$$2\mu_m V'_Y + S(B_{XX} + B_{ZZ}) - \rho_m g B\cos\Phi(1 + \gamma\Delta T) - p' = 0,$$

$$k_m T'_Y = q(T'_s + B\bar{T}_Y).$$

Note that the two equations in (A2) have been combined with the help of the continuity equation.

The nondimensionalization of length, time, velocity, pressure, and temperature respectively by d, d^2/κ_m, κ_m/d, $\rho_m\kappa_m^2/d^2$ and ΔT then reduces the above conditions to (7) given in Section 2.

INSTABILITE DUE A LA TENSION INTERFACIALE POUR UN FILM LIQUIDE COULAND LE LONG D'UN PLAN INCLINE

Résumé—On étudie par la théorie linéaire la circulation stationnaire, pilotée par la tension superficielle, dans un film mince coulant sur un plan incliné Dans le cas limite d'un film liquide horizontal, la circulation apparait à des nombres critiques finis d'onde et de Marangoni pour une lare étendue des paramètres d'écoulement si l'effet stabilisateur de la pesanteur, négligé par les chercheurs, est pris en compte. Si l'angle d'inclinaison θ est tel que $\sin\theta \ll 1$ et si le nombre de Rayleigh est de l'ordre de $\sin^3\theta$, la structure d'écoulement correspondant à l'état de stabilité neutre est celle de tourbillons orientés dans le sens de l'écoulement. Le nombre critique de Marangoni du film sur le plan incliné est plus petit que celui du film horizontal d'une valeur $\sin^2\theta F$, où F dépend des paramètres typiques de l'écoulement.

DURCH OBERFLÄCHENSPANNUNG ERZEUGTE INSTABILITÄT EINES AN EINER BEHEIZTEN, GENEIGTEN FLÄCHE HERABSTRÖMENDEN FLÜSSIGKEITSFILMS

Zusammenfassung—Mit einer linearen Theorie wird das Einsetzen der durch die Oberflächenspannung erzeugten stationären Zirkulationsströmung eines dünnen Films unter sucht, der an einer beheizten, geneigten Fläche herabströmt. Im Grenzfall eines horizontalen Flüssigkeitsfilms beginnt das Einsetzen der Zirkulationsströmung bei endlichen kritischen Wellenzahlen und Marangoni-Zahlen für einen großen Bereich wichtiger Strömungsparameter, wenn die stabilisierende Wirkung der Schwerkraft, welche bei früheren Autoren vernachlässigt wurde, berücksichtigt wird. Wenn der Neigungswinkel ϕ der Fläche so klein ist, daß $\sin\phi \ll 1$ und wenn die Rayleigh-Zahl von der Größenordnung $\sin^3\phi$ ist, dann hat die dem neutral stabilen Zustand entsprechende Strömung die Struktur von in Strömungsrichtung orientierten Wirbeln. Die kritische Marangoni-Zahl des Films an einer beheizten, geneigten Fläche ist um $\sin^2\phi F$ kleiner als die des entsprechenden horizontalen Films, wobei F von wichtigen Strömungsparametern abhängt.

НЕУСТОЙЧИВОСТЬ ТЕЧЕНИЯ ЖИДКОЙ ПЛЁНКИ НА НАГРЕТОЙ НАКЛОННОЙ ПОВЕРХНОСТИ, ВЫЗВАННАЯ СИЛАМИ ПОВЕРХНОСТНОГО НАТЯЖЕНИЯ

Аннотация — С помощью линейной теории исследуется возникновение устойчивого циркуляционного течения, вызванного силами поверхностного натяжения, при стекании тонкой пленки по наклонной нагретой поверхности. Найдено, что в предельном случае горизонтальной жидкой пленки циркуляционное течение может наблюдаться при конечных критических волновых числах и числах Марангони в широком диапазоне соответствующих параметров течения, если учитывается стабилизирующее влияние гравитации, которая пренебрегалась предыдущими исследователями. При малых углах наклона ($\sin \phi \ll 1$) и числах Релея порядка $\sin^3 \phi$ структура потока, соответствующая нейтрально устойчивому состоянию, представляет собой ориентированные по потоку вихри. Значение критического числа Марангони для пленки на нагретой наклонной поверхности меньше значения для горизонтальной пленки на величину $\sin^2 \phi F$, где F является функцией соответствующих параметров течения.